Optimal Contest Design: Tuning the Heat

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Introduction

- Many economic interactions can be described as contests:
    - promotions;
    - elections;
    - university entrance exams;
    - innovation competitions;
    - sporting events.
- All of these contests are designed.
- How then should contests be optimally designed?

Introduction

- The usual approach:
  - pick a contest family (e.g. Tullock, Lazear-Rosen, All-Pay),
  - then optimize (usually over prize vectors).
- Intuition we get often does not transfer across families:
  - Tullock $\longrightarrow$ winner-take-all is optimal (Clark and Riis, 1998; Schweinzer and Segev, 2012);
  - All-Pay $\longrightarrow$ $n-1$ equal prizes are optimal (Fang, Noe and Strack, 2020).
- Should we choose Tullock, All-Pay, or some other contest?

This paper

- Provides a general framework where the designer can choose
  - any prize profile and
  - any prize allocation rule (i.e., contest success function),

  including all standard contests as special cases.

- Focuses on the maximization of total effort net of prizes.

- New results:
  - risk-averse agents,
  - imperfect observability of effort.

- Extensions to heterogeneous agents, costly entry, and risk-loving agents.

Model

Environment

- A principal organizes a contest among $n \geq 2$ agents.
- In a contest, each agent
  - chooses an effort $e_i \geq 0$, and
  - obtains a monetary transfer $t_i \geq 0$.
- The payoff of agent i is

$$\Pi_i(e_i, t_i) = u(t_i) - c(e_i).$$

  - $u' > 0$, $u'' \leq 0$, and $u(0) = 0$;
  - $c' > 0$, $c'' > 0$, $c(0) = c'(0) = 0$, and $\lim_{e \to \infty} c'(e) = \infty$.

Environment

- Let $e = (e_1, \ldots, e_n)$, $E = \mathbb{R}_+^n$, and $t = (t_1, \ldots, t_n)$.
- The payoff of the principal is

$$\Pi_P(e,t) = \sum_{i=1}^{n} e_i - \sum_{i=1}^{n} t_i.$$

- Our results continue to hold with any production function $g : E \to \mathbb{R}_+$ that is symmetric, increasing and concave.

Environment

- After agents have chosen e, signal $s \in S$ is drawn according to some probability measure $\eta^e \in \Delta S$.
- (S, η) is the observational structure of the model.
- The principal observes s and not e.

- Examples
    - Perfect observability: $S = E$ and $s_i = e_i$.
    - Additive noise: $S = \mathbb{R}$ and $s_i = e_i + \epsilon_i$.
    - Correlated observational errors (Green and Stokey, 1983; Nalebuff and Stiglitz, 1983).
    - Various aggregate measures, e.g., $S = \mathbb{R}$ and $s = e_1 - e_2$.

Contests

- A contest (y, π) is defined by
  - a prize profile $y = (y_1, \ldots, y_n)$, w.l.o.g. $y_1 \geq \ldots \geq y_n$, and
  - a contest success function (CSF) $\pi : S \to \Delta T(y)$,

  where $T(y)$ is the set of all permutations of y.

- Given a fixed observational structure (S, η) and a contest (y, π), the probability that agent i wins the prize y_k is

$$p_i^k(e).$$

Examples

- Perfect observability, all-pay contest:

$$p_i^1(e) = \begin{cases} 1 & \text{if } e_i > e_j, \\ 1/2 & \text{if } e_i = e_j, \\ 0 & \text{if } e_i < e_j. \end{cases}$$

- Perfect observability, Tullock contest with impact function f:

$$p_i^1(e) = \begin{cases} \frac{f(e_i)}{f(e_i)+f(e_j)} & \text{if } \max\{e_i, e_j\} > 0, \\ 1/2 & \text{otherwise.} \end{cases}$$

- Imperfect observability with $s_i = e_i + \epsilon_i$, ϵ_i i.i.d. Gumbel with mean zero, and all-pay contest then for some $\beta > 1$:

$$p_i^1(e) = \frac{exp(e_i/\beta)}{exp(e_i/\beta) + exp(e_j/\beta)}$$

Principal's objective

- Let (possibly random) efforts be $\sigma_i \in \Delta\mathbb{R}_+$.
- The principal solves

$$\max_{\sigma,(y,\pi)} \mathbb{E}_\sigma\left[\sum_{i=1}^{n} e_i\right] - \sum_{i=1}^{n} y_i$$

  such that (y, π) implements σ.

Optimal Contest

Optimal prize vector and effort

- From Letina, Liu, Netzer (2020) we know that with perfect observability the optimal
  - prize vector y^* is:

$$y^* = \left(\frac{x^*}{n-1}, \ldots, \frac{x^*}{n-1}, 0 \right),$$

  - effort e^* is:

$$c(e^*) = \frac{n-1}{n} u\left(\frac{x^*}{n-1} \right),$$

  - where total sum x^* is:

$$u'\left(\frac{x^*}{n-1} \right) = c'\left(c^{-1}\left(\frac{n-1}{n} u\left(\frac{x^*}{n-1} \right) \right) \right).$$

Optimal contest

- Denote the probability that agent i wins one of the top $n-1$ prizes with $p_i^{-n}(e) = 1 - p_i^n(e)$.

Proposition 1

Fix an arbitrary observational structure (S, η). A contest (y, π) is optimal if the prize profile is $y = y^*$ and the CSF satisfies, for each $i \in I$,

(i) $p_i^{-n}(e^*, e_{-i}^*) = \frac{n-1}{n}$, and

(ii) $p_i^{-n}(e_i, e_{-i}^*) \leq \frac{c(e_i)}{u(x^*/(n-1))}, \ \forall e_i \neq e^*$.

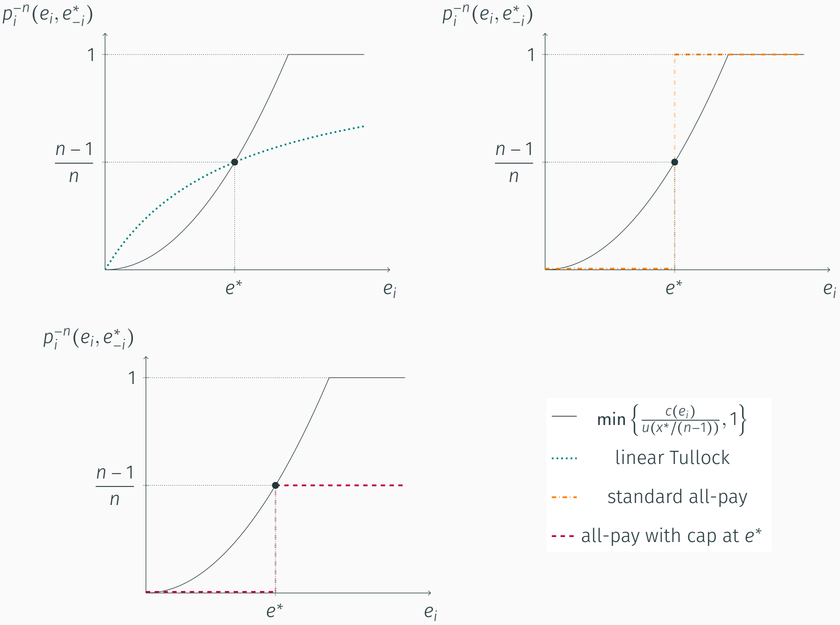

Figure 1: $n = 2$, $u(t) = \sqrt{t}$ and $c(e) = e^2$.

Perfect Observability of Effort

Nested Tullock contests

- Nested Tullock was introduced by Clark and Riis (1996).
- With n agents and a single positive prize, the probability that i wins the prize is:

$$p_i(e) = \frac{f(e_i)}{\sum_{j=1}^{n} f(e_j)}. \tag{1}$$

- With multiple positive prizes, (1) is applied in a nested fashion by eliminating the winners in each round sequentially.

Nested Tullock contests

Proposition 2

Suppose efforts are perfectly observed. Then, the nested Tullock contest is optimal if the prize profile is $y = y^*$ and the CSF is a nested Tullock with

$$f(e_i) = c(e_i)^{r^*(n)} \quad \text{and} \quad r^*(n) = \frac{n-1}{H_n - 1},$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number.

Optimal Tullock: effort and competitiveness

- With perfect observability and risk-averse agents, optimal contest achieves second-best

$$e^* < e^{FB}$$

  but efficiency loss vanishes as $n \to +\infty$.

- The precision of the CSF, r^*, measures competitiveness:
  - $r^*(2) = 2$, $r^*(n) \uparrow$ in n, and $\lim_{n\to\infty} r^*(n) = \infty$
  - $r^*(n)$ is such that any increase in the competitiveness of the contest would destroy the pure strategy equilibrium.

Optimal Tullock: results from the literature

- Take a winner-take-all all-pay contest.
- Fang, Noe and Strack (2020) show that "turning down the heat" by dividing the prize increases the total expected effort.
- They conclude that the optimal all-pay contest has $n-1$ equal prizes.
- We show that it is beneficial to turn down the heat even further by making the CSF less precise.
- Schweinzer and Segev (2012) show that turning up the heat (by making the prize profile more top-heavy) is beneficial as long as a pure strategy equilibrium exists.
- The optimal "competitiveness" of the contest is exactly at the point where the pure strategy equilibrium appears.

Imperfect Observability of Effort

Imperfect observability of effort

- (S, η) features symmetric additive noise if
  - $s_i = e_i + \varepsilon_i$,
  - ε_i are i.i.d.,
  - from cdf F and support contained in $[\underline{\varepsilon}, \overline{\varepsilon}]$.

Proposition 3

Suppose efforts are observed with symmetric additive noise. If

$$F^{-}(\underline{\varepsilon} + e^* - e) \geq 1 - \frac{c(e)}{c(e^*)}, \quad \forall e \in [0, e^*],$$

then a contest with prize profile $y = y^*$ and an all-pay allocation rule with a cap at $\bar{s} = e^* + \underline{\varepsilon}$ is optimal.

- Examples in the paper for multiplicative noise and observation of $e_1 - e_2$.

Imperfect observability of effort

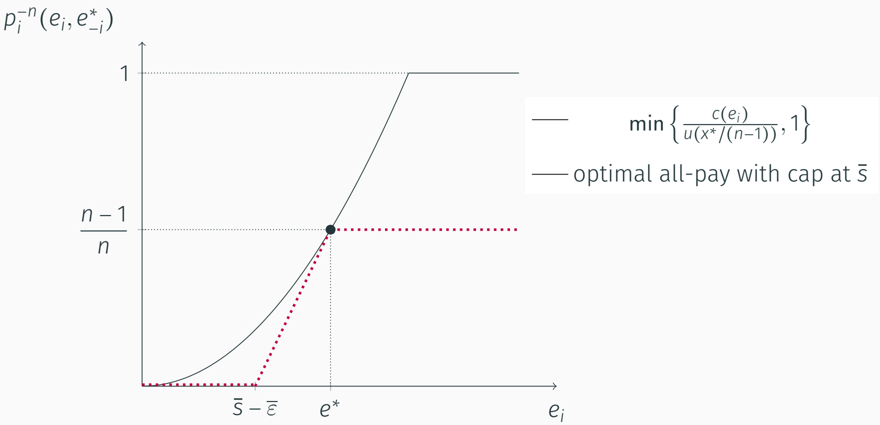

Figure 2: $\varepsilon_i \sim U[-0.1, 0.1]$, $n = 2$, $u(t) = \sqrt{t}$ and $c(e) = e^2$.

Extensions

Extensions

- Heterogeneous contestants
  - $n = 2$: biased Tullock is optimal for arbitrary cost functions.
  - $n > 2$: $n - 1$ equal positive prizes and one zero prize are optimal if heterogeneity is not too large.
- Costly entry, with private cost
  - $n - 1$ equal positive prizes, with last prize potentially positive.
- Risk-loving agents
  - WTA is optimal,
  - otherwise, Prop. 1 carries over.

Concluding remarks

Contributions

- We provide a framework that enables us to study contest design, without being restricted to a single class of contests.
- We provide sufficient conditions for a contest to be optimal for an arbitrary observational structure (S, η).
- With perfect observability, we show the optimum can be achieved by an appropriately designed Tullock contest.
- With imperfect observability and symmetric additive noise, we provide sufficient conditions on the noise distribution and describe an optimal contest if those conditions are satisfied.

Open questions

- We focus on optimal design that maximizes aggregate effort. But there are other objectives that the principal may have. The immediate one is maximizing highest effort.
- We focus on observational structures for which the second-best is implementable. How does the optimal contest look like when this is not the case? What is the third-best?
  - Conjecture: $n-1$ positive equal prizes is no longer optimal, prizes are more concentrated at the top.