B Online Appendix — Not for Publication

The Road Not Taken: Competition and the R&D Portfolio

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In this Appendix I consider four extensions of the basic model. First two deal with the possibility that a merger could generate efficiencies which could overturn the result that a merger decreases the variety of approaches to innovation. In Section B.2 I consider a mixed strategy equilibirum and show that the equilibrium structure and determinants of comparative static results found for pure strategy equilibria are robust. Finally, in Section B.3 I consider the case when the research budgets are limited or when financing of research is costly. Proofs are presented sequentially in the end of the Appendix.

B.1 Efficiency defense

General cost reduction

Consider the original setting, but suppose that if two firms merge, they become more efficient at developing innovations. That is, suppose that for the merged firm the fixed cost of developing any given approach j is given by $\tilde{C}(j;\epsilon) : [0,1) \to \mathbb{R}^+$ such that $\tilde{C}(j;\epsilon) \leq C(j)$ for all j. Like C, assume that $\tilde{C}$ is continuous, differentiable, strictly increasing and that $\lim_{j\to 1} \tilde{C}(j;\epsilon) = \infty$. Finally, suppose that in this setting ϵ captures the size of the efficiency gains resulting from the merger, such that $\partial\tilde{C}(j;\epsilon)/\partial\epsilon < 0$. Simple functional forms that satisfy these assumptions (for the appropriate domain of ϵ) are: (i) additive $\tilde{C}(j;\epsilon) = C(j) - \epsilon$ and (ii) multiplicative $\tilde{C}(j;\epsilon) = (1-\epsilon)C(j)$. Let the number of symmetric firms with innovation cost functions $C(j)$ in the pre-merger market be N. Suppose that after the merger, the merged firm has the cost function $\tilde{C}(j;\epsilon)$ and that the remaining $N-2$ firms are active with cost functions $C(j)$.

In this setting, if the efficiency gain from the merger is sufficiently large, there will be no loss of diversity in the approaches to innovation as a consequence of the merger.

Proposition 7 (Merger with general cost reductions).

Suppose that Assumptions 1, 2, and 3 hold and that the merger results in efficiency gains as above. Then:

1. *A PSE in the post-merger market always exists.*
2. *In any PSE in the post-merger market the set of developed approaches is $[0, \tilde{\alpha}_1)$, where $\tilde{\alpha}_1$ is given by $\tilde{C}(\tilde{\alpha}_1; \epsilon) = R(1) - r(0, N-1)$.*
3. *If the technology frontier in the market without a merger is given by α_1, then the merger does not reduce the variety of approaches to innovation if and only if*

$$C(\alpha_1) - \tilde{C}(\alpha_1; \epsilon) \geq r(0, N-1) - r(0, N). \tag{4}$$

From Proposition 3 we know that a merger, via the Arrow effect, reduces the incentives to invest. This is captured by the right-hand side of the inequality (4). However, Proposition 7 states that if the efficiency caused by the merger is large enough, which is given by the left-hand side of the inequality (4), it can outweigh the decrease in the incentive to invest. In this case, the merger does not lead to a decrease in the variety of approaches to innovation. In the case of additive efficiency gains, that is if $\tilde{C}(j; \epsilon) = C(j) - \epsilon$, the inequality (4) would simplify to $\epsilon \geq r(0, N-1) - r(0, N)$. In the case of multiplicative efficiency gains, that is if $\tilde{C}(j; \epsilon) = (1 - \epsilon)C(j)$, the inequality (4) simplifies to $\epsilon C(\alpha_1) \geq r(0, N-1) - r(0, N)$. It is clear that there always exists ϵ large enough such that these inequalities are satisfied, and that such a merger would not lead to a decrease in the variety of approaches to innovation.

Approach-specific synergies

In the previous section, I considered a situation in which a merger between any two firms leads to the same efficiency gains. Now, suppose that each firm has some specific knowledge and that if the two firms merged, they could combine this specific knowledge in a way that would enable the merged entity to conduct research over some specific interval of approaches more efficiently. In this setting, it will not only be the size of the efficiency gains that will be required for a successful efficiency defense, but also that the efficiency gains occurs over approaches that would not have been developed in the post-merger market absent the efficiency gains.

For concreteness, consider this simple extension of the model. Suppose that each firm $i \in \{1, \ldots, N\}$ is located on the unit line in an equidistant manner. That is, the location of the firm is given by $i/(N+1)$. Firm's location represents its specific knowledge. On it its own, this knowledge is worthless. However, suppose that firm i merged with some firm $l \in \{1, \ldots, N\}$, $i \neq l$. Then the merged entity would receive efficiency gains over an interval midway between the location of the firms i and l. That is, the merged entity is more efficient over an interval

$$\left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)$$

for some $0 < \delta \leq 1/(N+1)$.[17] For simplicity, suppose that on the above interval the cost of developing an approach is zero for the merged entity. That is, a firm which has not merged has the innovation cost function $C(j)$ and the merged firm (where the merging firms are i and l) has the innovation cost function

$$\tilde{C}_{i,l}(j) = \begin{cases} 0 & \text{if } j \in \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right) \\ C(j) & \text{otherwise} \end{cases}.$$

In this setting, a merger will not lead to a decrease in the variety of approaches if the efficiency gain covers large enough interval (δ is large enough) and if the efficiency gain occurs over projects which would not be developed absent the efficiency gain. The latter depends on which firms actually merge. Thus, for the same size of the efficiency gain from the merger, some mergers will lead to a decrease in the variety of approaches whereas others will not.

Proposition 8 (Merger with approach-specific synergies)**.**

Suppose that Assumptions 1, 2, and 3 hold and that the merger results in efficiency gains

[17]The upper bound is a simplification that ensures that efficiency gains are always in the unit interval. It would be straightforward to remove it, at the cost of more cumbersome notation.

as above. Then, if firms i and l merge:

1. *A PSE in the post-merger market always exists.*

2. *In any PSE in the post-merger market the set of developed approaches is* $[0, \tilde{\alpha}_1) \cup \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)$, *where* $\tilde{\alpha}_1$ *is given by* $C(\tilde{\alpha}_1) = R(1) - r(0, N-1)$.

3. *If the technology frontier in the market without a merger is given by* α_1, *then the merger does not reduce the variety of approaches to innovation if and only if*

$$\int_{x \in [0,\tilde{\alpha}_1) \cup \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)} dx \geq \alpha_1. \tag{5}$$

As the innovation cost function for the merged firm is not strictly increasing, the set of developed approaches need not be convex any more. However, the intuition is clear — the efficiency gain must be both large enough and must materialize over the projects which would not have been developed otherwise for the efficiency defense to be successful.

B.2 Mixed strategies

Consider the original setting, but suppose that firms are using mixed strategies. As a simplifying assumption, I will consider only the following pure strategy space

$$\mathcal{I}^m = \{0\} \cup \Big\{[0, j) : j \in (0, 1)\Big\}$$

and I will look only at symmetric mixed strategy equilibria (SMSE). Because now the pure strategy of a firm is restricted to choosing an interval $[0, j)$, it can be identified with the upper bound of the interval j. Denote with $f_i(j)$ the density that the firm i chooses the interval $[0, j)$ and with $F_i(j)$ the related cumulative distribution function.

Proposition 9 (Characterization of SMSE)**.** *Suppose $N = 2$ and the Assumptions 1 and*

2 hold. Then the unique SMSE is characterized by the cumulative distribution function:

$$F(j) = \begin{cases} 0 & \text{if} \quad \dfrac{C(j) - R(2)}{R(1) - r(0,2) - R(2)} < 0 \\ \dfrac{C(j) - R(2)}{R(1) - r(0,2) - R(2)} & \text{if} \quad \dfrac{C(j) - R(2)}{R(1) - r(0,2) - R(2)} \in [0,1] \\ 1 & \text{if} \quad \dfrac{C(j) - R(2)}{R(1) - r(0,2) - R(2)} > 1 \end{cases}$$

for $j \in [0, 1)$.

Suppose that $R(2) - C(0) > 0$. In pure actions, by Proposition 2 it holds: $m = 2$, $C(\alpha_1) = R(1) - r(0, N)$ and $C(\alpha_2) = R(2)$. Thus, both firms will invest in the interval $[0, \alpha_1)$, only one firm will invest in the interval $[\alpha_1, \alpha_2)$ and no firm will invest in $[\alpha_2, 0)$. Now consider SMSE. By Proposition 9, for $j \in [0, \alpha_1)$ it holds $F(j) = 0$, thus both firms invest in this interval with probability 1. For $j \in (\alpha_1, \alpha_2)$ it holds $0 < F(j) < 1$, thus firms invest with some probability less then one. If $j \in [\alpha_2, 0)$, then $F(j) = 1$, so that firms do not invest in this interval. Similar results hold if $R(2) - C(0) \leq 0$. Thus, the basic structure of the model is the same in both pure and mixed strategy equilibria. In particular the k-firm frontiers are the same. Furthermore, comparative statics results regarding variety of projects undertaken remain qualitatively the same, as anything that affects the one-firm frontier has qualitatively the same effect both in pure action and in mixed strategy equilibria. Figure 8 illustrates the difference between the (expected) equilibrium market portfolios for the Cournot duopoly example from the appendix A.10. The mixed strategy equilibrium is "smoother" than the pure strategy equilibrium. The reason for this is that the integer problem is not present in the mixed strategy setting. In pure strategy equilibrium, some projects have higher expected profits than others (i.e. project $\alpha_2 + \epsilon$ is more profitable than $\alpha_2 - \epsilon$ for some small positive ϵ). In mixed strategy equilibria, all projects in the interval where the mixing occurs have the same expected profits.

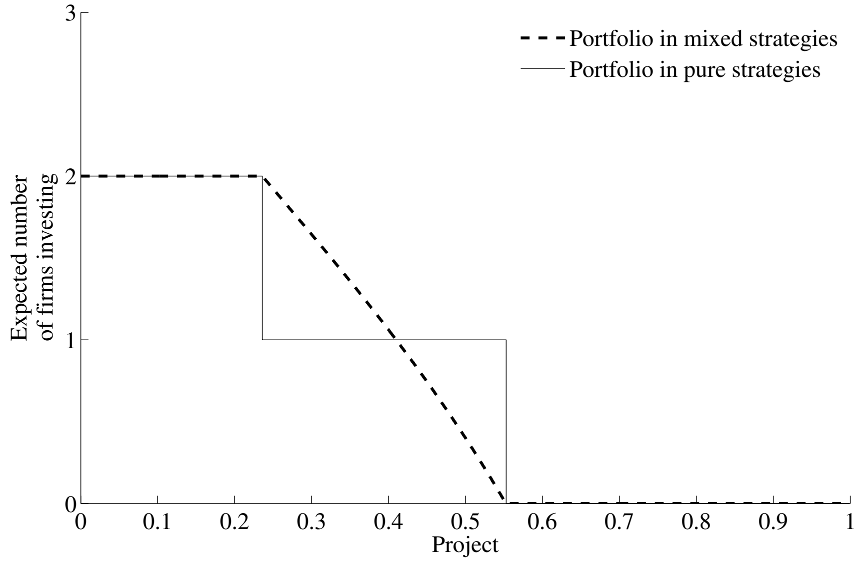

Figure 8: Symmetric mixed strategy equilibrium.

B.3 Limited budget and costly financing

This section considers the case where firms face an exogenous constraint on their research budgets. This constraint can take the form of a budget constraint, or it can (equivalently) take the form of costly financing for research. The main result is that a binding budget constraint or a costly source of financing imposes a positive opportunity cost on investments in research projects, but that the main mechanics of the model remain unchanged.

First, suppose that there are two firms in a market and that each firm has a budget B and suppose that the budget is binding, in the sense that firms would want to invest more in research if they had more resources.[18] Then the following result is obtained:

Proposition 10 (Equilibrium in a game with limited budget)**.**
Suppose that Assumptions 1 and 2 hold, and that there are two firms with a budget B. Then, a PSE always exists, the induced PSE market portfolio is unique and any investment plan which induces a portfolio identical to the market PSE portfolio is itself a PSE. Furthermore, there exists a unique $\beta > 0$ such that:

1. *the maximum number of firms investing in any project m^b is given by*

$$m^b = \max_{\{1,2\}} n \qquad s.t. \qquad R(n) - r(n-1, N) - C(0) > \beta.$$

[18]Formally, if $m = 2$ then $2B < 2\int_0^{\alpha_2} C(j)dj + \int_{\alpha_2}^{\alpha_1} C(j)dj$ and if $m = 1$ then $2B < \int_0^{\alpha_1} C(j)dj$.

2. Firm frontiers are determined by

$$R(1) - r(0, N) - C(\alpha_1^b) =$$
$$R(m^b) - r(m^b - 1, N) - C(\alpha_m^b) = \beta.$$

3. Let $\alpha_{m+1}^b = 0$ and $\alpha_0^b = 1$. The total expenditure is

$$m \int_0^{\alpha_m^b} C(j)dj + (m-1) \int_{\alpha_m^b}^{\alpha_{m-1}^b} C(j)dj = 2B.$$

Then the PSE portfolio $n^b(j)$ is given by

$$n^b(j) = k \quad \text{if} \quad j \in [\alpha_{k+1}^b, \alpha_k^b).$$

As can be seen from conditions 1. and 2., the basic form of the market equilibrium portfolio will remain unchanged. The only difference is that the budget constraint will impose positive opportunity cost β on the choice of research projects, as opposed to the unconstrained equilibrium where the opportunity costs was 0. In the scenario where firms can borrow unlimited funds at some positive price, the equilibrium characterized above still holds, but now β is exogenously given and as a function of the cost of financing.

B.4 Proof of Proposition 7

I prove each of the three statements contained in Proposition 7 in turn. The proof is analogous to the proof of Proposition 1.

Lemma 7 (Existence)**.** *A PSE in the post-merger market always exists.*

I provide a constructive proof of Lemma 7 in three steps. Step 1 constructs the candidate equilibrium investment plan $\tilde{I}$. Step 2 proves that no firm can increase its expected profits by making additional investments. Step 3 proves that no firm can increase its expected profits by reducing investments. Finally, notice that any deviation from the investment plan $\tilde{I}$ can be written as a collection of investments and divestments and by Steps 2 and 3, each such investment and divestment decreases expected profits and hence any such collection must decrease expected profits. Thus, no firm can profitably deviate from the investment plan $\tilde{I}$ and then, by definition, $\tilde{I}$ is an equilibrium.

Step 1. *Constructing the candidate equilibrium.*

Given a game, define m such that

$$m = \max_{\{1,\dots,N-1\}} n$$

$$\text{s.t. } R(n) - r(n-1, N-1) - C(0) > 0$$

As by assumption $R(1) - r(0, N-1) - C(0) > 0$, a solution to this maximization problem always exists.

Next, calculate each $\tilde{\alpha}_1, \alpha_2, \dots, \alpha_m$ such that the following condition holds:

$$\begin{aligned} R(1) - r(0, N-1) - \tilde{C}(\tilde{\alpha}_1; \epsilon) &= \\ R(2) - C(\alpha_2) &= \\ R(3) - C(\alpha_3) &= \\ &\vdots \\ R(m) - C(\alpha_m) &= \ 0. \end{aligned}$$

By construction it holds $R(m) - r(m-1, N) - C(0) > 0$ and by Assumption 1 the reward of innovation are non-increasing, so the inequality holds for all $k < m$. As $\tilde{C}(j; \epsilon) \leq C(j)$, the inequality also holds for the merged firm. As costs of innovation approach infinity as $j \to 1$, values $\tilde{\alpha}_1, \alpha_2, \dots, \alpha_m$ always exist by the Intermediate Value Theorem. Furthermore, as $C(j)$ is increasing, $\tilde{C}(j; \epsilon) \leq C(j)$, and by applying Assumption 1 it is easy to see that $\tilde{\alpha}_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$.

Observe that $N - 1 \geq m$. Label the merged firm with subscript $i = 1$ and all other firms with $i \in \{2, \dots, N-1\}$. For the merged firm, let $\tilde{I}_1 = [0, \tilde{\alpha}_1)$. For each $i \in \{2, \dots, m\}$, let $\tilde{I}_i = [0, \alpha_i)$. For each $i \in \{m+1, \dots, N-1\}$ let $\tilde{I}_i = \emptyset$. I will demonstrate that $\tilde{I}$ is an equilibrium.

Step 2. *Suppose that $\tilde{I}$ is constructed as above. Then no firm can increase its expected* *profits by making additional investments.*

Proof. First observe that for all firms $i \in \{2, \ldots, N-1\}$ the argument is identical as in the proof of Proposition 1, as the investment decision of the firm only depends on their investment costs and the number of firms investing in any given project. Thus we only need to show that the merged firm cannot increase profits by making additional investments. This holds by construction. The merged firm already invests in the entire interval $[0, \tilde{\alpha}_1)$. For any $j > \tilde{\alpha}_1$ it holds $R(1) - r(0, N-1) - \tilde{C}(j; \epsilon) < 0$ as $\tilde{C}(j; \epsilon)$ is strictly increasing in j. Thus, no additional profitable investments exist for the merged firm. □

Step 3. *Suppose that $\tilde{I}$ is constructed as above. Then no firm can increase its expected profits by decreasing investments.*

Proof. Similar to the argument in the previous step, it is sufficient to show that the merged firm cannot increase profits by decreasing investments. First, observe that for $j \in [\alpha_2, \tilde{\alpha}_1)$ the investment is profitable as it holds $R(1) - r(0, N-1) - \tilde{C}(j; \epsilon) > 0$ for all j in the interval. For all j in $[0, \alpha_2)$ it holds $R(n(j, \tilde{I})) - C(j) > 0$ (otherwise non-merged firms would have an incentive to divest) and as $\tilde{C}(j; \epsilon) \leq C(j)$, it also holds $R(n(j, \tilde{I})) - \tilde{C}(j; \epsilon) > 0$ for all $j \in [0, \alpha_2)$. Hence, the merged firm cannot increase profits by divesting. □

Lemma 8. *In any PSE in the post-merger market the set of developed approaches is $[0, \tilde{\alpha}_1)$, where $\tilde{\alpha}_1$ is given by $\tilde{C}(\tilde{\alpha}_1; \epsilon) = R(1) - r(0, N-1)$.*

Proof. Suppose not. Then, either there exist an interval $l \subseteq [0, \tilde{\alpha}_1)$ where no firm invests, or there exists an interval $l' \subseteq [\tilde{\alpha}_1, 1)$ where at least one firm invests, or both. First suppose that an interval l exists. As $\tilde{C}(j; \epsilon)$ is strictly increasing, then for all $j \in l$ it holds $\tilde{C}(j; \epsilon) < R(1) - r(0, N-1)$. Hence the merged firm can profitably invest in the subset of l. Next, suppose an interval l' exists. Observe that for any $j > \tilde{\alpha}_1$ it holds $C(j) \geq \tilde{C}(j; \epsilon) > R(1) - r(0, N-1)$. By Assumption 1 it then also holds $C(j) \geq \tilde{C}(j; \epsilon) > R(k)$ for all $k \geq 2$. Thus, no firm can profitably invest any subset of l'. □

Lemma 9. *If the technology frontier in the market without a merger is given by α_1, then the merger does not reduce the variety of approaches to innovation if and only if*

$$C(\alpha_1) - \tilde{C}(\alpha_1; \epsilon) \geq r(0, N-1) - r(0, N).$$

Proof. The merger does not reduce variety if and only if $\tilde{\alpha}_1 \geq \alpha_1$. As $\tilde{C}(j; \epsilon)$ is strictly increasing, this will hold if and only if $\tilde{C}(\alpha_1; \epsilon) \leq R(1) - r(0, N-1)$. By Proposition 2, we know that $C(\alpha_1) = R(1) - r(0, N)$. Subtracting the above inequality, the claim follows. □

B.5 Proof of Proposition 8

I prove each of the three statements contained in Proposition 8 in turn. The proof is analogous to the proof of Proposition 1.

Lemma 10 (Existence)**.** *A PSE in the post-merger market always exists.*

I provide a constructive proof of Lemma 10 in three steps, analogous to the proof of Lemma 7.

Step 1. *Constructing the candidate equilibrium.*

Given a game, define m such that

$$m = \max_{\{1,\ldots,N-1\}} n$$

$$\text{s.t. } R(n) - r(n-1, N-1) - C(0) > 0$$

As by assumption $R(1) - r(0, N-1) - C(0) > 0$, a solution to this maximization problem always exists.

Next, calculate each $\tilde{\alpha}_1, \alpha_2, \ldots, \alpha_m$ such that the following condition holds:

$$\begin{aligned} R(1) - r(0, N-1) - C(\tilde{\alpha}_1) &= \\ R(2) - C(\alpha_2) &= \\ R(3) - C(\alpha_3) &= \\ &\vdots \\ R(m) - C(\alpha_m) &= \ 0. \end{aligned}$$

By construction it holds $R(m)-r(m-1,N)-C(0) > 0$ and by Assumption 1 the reward of innovation are non-increasing, so the inequality holds for all $k < m$. As costs of innovation approach infinity as $j \to 1$, values $\tilde{\alpha}_1, \alpha_2, \ldots, \alpha_m$ always exist by the Intermediate Value Theorem. Furthermore, by Assumption 1, it is easy to see that $\tilde{\alpha}_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$.

Observe that $N - 1 \geq m$. Label the merged firm with subscript $i = 1$ and all other firms with $i \in \{2, \ldots, N-1\}$. For the merged firm, let $\tilde{I}_1 = [0, \tilde{\alpha}_1) \cup \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)$. For each $i \in \{2, \ldots, m\}$, let $\tilde{I}_i = [0, \alpha_i)$. For each $i \in \{m+1, \ldots, N-1\}$ let $\tilde{I}_i = \emptyset$. I will demonstrate that $\tilde{I}$ is an equilibrium.

Step 2. *Suppose that $\tilde{I}$ is constructed as above. Then no firm can increase its expected profits by making additional investments.*

Proof. For firms $i \in \{2, \ldots, N-1\}$ the argument is identical as in the proof of Proposition 7. Thus we only need to show that the merged firm cannot increase profits by making additional investments. The merged firm already invests in the entire set $[0, \tilde{\alpha}_1) \cup \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)$. For any $j > \tilde{\alpha}_1$ and $j \notin \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)$ it holds $R(1) - r(0, N-1) - \tilde{C}(j) < 0$ as $C(j)$ is strictly increasing. Thus, no additional profitable investments exist for the merged firm. □

Step 3. *Suppose that $\tilde{I}$ is constructed as above. Then no firm can increase its expected profits by decreasing investments.*

Proof. Similar to the argument in the previous step, it is sufficient to show that the merged firm cannot increase profits by decreasing investments. First, observe that for $j \in [\alpha_2, \tilde{\alpha}_1)$ the investment is profitable as it holds $R(1)-r(0,N-1)-C(j) > 0$ for all j in the interval. For all j in $[0, \alpha_2)$ it holds $R(n(j, \tilde{I})) - C(j) > 0$ (otherwise non-merged firms would have an incentive to divest) and as $\tilde{C}(j) \leq C(j)$, it also holds $R(n(j, \tilde{I})) - \tilde{C}(j) > 0$ for all $j \in [0, \alpha_2)$. For all $j \in \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)$ the investment is costless. Hence, the merged firm cannot increase profits by divesting. □

Lemma 11. *In any PSE in the post-merger market the set of developed approaches is* $[0, \tilde{\alpha}_1) \cup \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)$, *where* $\tilde{\alpha}_1$ *is given by* $C(\tilde{\alpha}_1) = R(1) - r(0, N-1)$.

Proof. Suppose not. Then, either there exists an interval $l \subseteq [0, \tilde{\alpha}_1)$ or an interval $l' \subseteq \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)$ where no firm invests, or there exists an interval $l'' \subseteq [0,1) \setminus \left([0, \tilde{\alpha}_1) \cup \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)\right)$ where at least one firm invests. First suppose that an interval l exists. Because by construction it holds $C(j;) < R(1) - r(0, N-1)$ for all $j < \tilde{\alpha}_1$, any firm can profitably invest in the set l. Next, suppose l' exists. The merged firm can invest in the set l' without any cost, hence it can increase its expected profit by investing. Finally, suppose an interval l'' exists. Observe that for any $j > \tilde{\alpha}_1$ and $j \notin \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)$ it holds $\tilde{C}(j) > R(1) - r(0, N-1)$. By Assumption 1 it then also holds $C(j) \geq \tilde{C}(j) > R(k)$ for all $k \geq 2$. Thus, no firm can profitably invest any subset of l''. □

Lemma 12. *If the technology frontier in the market without a merger is given by* α_1, *then the merger does not reduce the variety of approaches to innovation if and only if*

$$\int_{x \in [0, \tilde{\alpha}_1) \cup \left[\frac{i+l}{2(N+1)} - \delta, \frac{i+l}{2(N+1)} + \delta\right)} dx \geq \alpha_1.$$

Proof. Without the merger, the set of developed approaches by Proposition 2 is $[0, \alpha_1)$. The result follows by Claim 2 of the Proposition. □

B.6 Proof of Proposition 9

Suppose, without loss of generality, that the firm 2 invests according to some probability density function f_2, with the cumulative density function F_2. Consider any pure action

x_1 of firm 1. The profit of the firm 1 can be expressed as:

$$\pi_1(x_1|F_2) = -\int_0^{x_1} C(j)dj + \int_0^{x_1} \left[\int_0^{x_2} R(2)dj + \int_{x_2}^{x_1} R(1)dj + \int_{x_1}^{1} r(0,2)dj\right] f_2(x_2)dx_2 + \\ + \int_{x_1}^{1} \left[\int_0^{x_1} R(2)dj + \int_{x_2}^{1} r(0,2)dj\right] f_2(x_2)dx_2.$$

Deriving:

$$\frac{d\pi_1(x_1|F_2)}{dx_1} = -C(x_1) + \left[\int_0^{x_1} R(2)dj + \int_{x_1}^{x_1} R(1)dj + \int_{x_1}^{1} r(0,2)dj\right] f_2(x_1) - \\ - \left[\int_0^{x_1} R(2)dj + \int_{x_1}^{1} r(0,2)dj\right] f_2(x_1) + \\ + \int_0^{x_1} [R(1) - r(0,2)]\, f_2(x_2)dx_2 + \int_{x_1}^{1} R(2) f_2(x_2)dx_2$$

and simplifying:

$$\frac{d\pi_1(x_1|F_2)}{dx_1} = -C(x_1) + [R(1) - r(0,2)]\, F_2(x_1) + R(2)\,(1 - F_2(x_1))\,.$$

Next, use the assumption that the equilibrium is symmetric, that is $F_1 = F_2 = F$. In equilibrium it has to hold $d\pi_1(x_1|F)/dx_1 = 0$ for all x_1 in the support of f. This condition is uniquely satisfied by

$$\tilde{F}(j) = \frac{C(j) - R(2)}{R(1) - r(0,2) - R(2)}$$

for j in the support of f.

Observe that $\tilde{F}$ is strictly increasing and, for all j such that $\tilde{F}(j) < 0$ it follows that $d\pi_1(x_1, F)/dx_1 > 0$, and for all j such that $\tilde{F}(j) > 1$ it follows that $d\pi_1(x_1, F_2)/dx_1 < 0$. Hence, the unique symmetric equilibrium is given by the profile (F, F) where

$$F(j) = \begin{cases} 0 & \text{if} \quad \dfrac{C(j) - R(2)}{R(1) - r(0,2) - R(2)} < 0 \\ \dfrac{C(j) - R(2)}{R(1) - r(0,2) - R(2)} & \text{if} \quad \dfrac{C(j) - R(2)}{R(1) - r(0,2) - R(2)} \in [0,1] \\ 1 & \text{if} \quad \dfrac{C(j) - R(2)}{R(1) - r(0,2) - R(2)} > 1 \end{cases}$$

for $j \in [0, 1)$.

B.7 Proof of Proposition 10

I prove this statement in three steps. First I show that some β satisfying all conditions in the proposition always exists and is unique. Next, I construct an investment plan inducing the same portfolio as the one in Proposition 10. Finally I show that the constructed investment plan is an equilibrium and that any investment plan inducing the same portfolio is an equilibrium as well.

Lemma 13. *β always exists and is unique.*

Proof. Define functions $\psi^1(\beta) : [0, \beta^1] \to \mathbb{R}^+$, $\psi^2(\beta) : [0, \beta^2] \to \mathbb{R}^+$ such that

$$\psi^1(\beta) = \int_0^{C^{-1}\left(R(1)-r(0,2)-\beta\right)} C(j)dj$$

$$\psi^2(\beta) = \int_0^{C^{-1}\left(R(2)-\beta\right)} C(j)dj + \int_0^{C^{-1}\left(R(1)-r(0,2)-\beta\right)} C(j)dj$$

with $\beta^1 = R(1) - r(0, 2) - C(0)$ and $\beta^2 = R(2) - C(0)$. As $C(\cdot)$ is continuous, strictly increasing and defined on an interval, its inverse is continuous and strictly increasing as well. Hence both $\psi^1(\beta)$ and $\psi^1(\beta)$ are continuous and strictly decreasing. Furthermore, by Assumption 1 it holds $\beta^1 \geq \beta^2$.

Either (i) $\psi^1(\beta^2) \geq 2B$ or (ii) $\psi^1(\beta^2) < 2B$. If (i) is true, $\psi^1(\beta^2) \geq 2B$ and $\psi^1(\beta^1) = 0 < 2B$. By the Intermediate Value Theorem there exists some $\beta^* \in [\beta^2, \beta^1)$ such that $\psi^1(\beta^*) = 2B$ and furthermore β^* is unique because $\psi^1(\beta)$ is strictly decreasing. Observe that $\beta^* \in [R(2) - C(0), \beta^1)$, hence $R(1) - r(0, 2) - C(0) > \beta^*$ and $R(2) - C(0) \leq \beta^*$. Thus, by the condition 1. of Proposition 10 we have $m^b = 1$. By the condition 2. the firm frontier is $\alpha_1^b = C^{-1}\big(R(1) - r(0, 2) - \beta^*\big)$. Finally, the condition 3. holds because $\int_0^{\alpha_1^b} C(j)dj = 2B$ by construction. Hence, β^* uniquely satisfies all three conditions of the Proposition 10.

If (ii) is true, then $\psi^2(\beta^2) < 2B$ and $\psi^2(0) > 2B$, by the assumption of the binding budget constraint. By the Intermediate Value Theorem there exists some $\beta^* \in (0, \beta^2)$ such that $\psi^2(\beta^*) = 2B$ and furthermore β^* is unique because $\psi^2(\beta)$ is strictly decreasing. Observe that $\beta^* \in (0, \beta^2)$, hence $R(2) - C(0) > \beta^*$. Thus, by the condition 1. of Proposition 10 we have $m^b = 2$. By the condition 2. the firm frontiers are $\alpha_1^b = C^{-1}\big(R(1) - r(0,2) - \beta^*\big)$ and $\alpha_2^b = C^{-1}\big(R(2) - \beta^*\big)$. Finally, the condition 3. holds because $\int_0^{\alpha_2^b} C(j)dj + \int_0^{\alpha_1^b} C(j)dj = 2B$ by construction. Hence, β^* uniquely satisfies all three conditions of the Proposition 10. □

Lemma 14. *An equilibrium inducing portfolio equivalent to the one characterized in Proposition 10 can always be constructed.*

Proof. Either $m = 2$ or $m = 1$. If $m = 1$, then it holds $\int_0^{\alpha_1^b} C(j)dj = 2B$. Then there exists a point x such that $0 < x < \alpha_1^b$ and $\int_0^x C(j)dj = B$ and $\int_x^{\alpha_1^b} C(j)dj = B$. Let one firm invest in the interval $[0, x)$ and the other firm in the interval $[x, \alpha_1^b)$. This investment plan generates a portfolio equivalent to the one characterized.

If $m = 2$, then it holds $2\int_0^{\alpha_2^b} C(j)dj + \int_{\alpha_2^b}^{\alpha_1^b} C(j)dj = 2B$. Then there exists a point x such that $\alpha_2^b \le x \le \alpha_1^b$ and $\int_0^x C(j)dj = B$ and $\int_0^{\alpha_2^b} C(j)dj + \int_x^{\alpha_1^b} C(j)dj = B$. Let one firm invest in the interval $[0, x)$ and the other firm in the set $[0, \alpha_2^b) \cup [x, \alpha_1^b)$. This investment plan generates a portfolio equivalent to the one characterized. □

Lemma 15. *The investment plan constructed in Lemma 14 is an equilibrium and any investment plan inducing the same portfolio is an equilibrium as well.*

Proof. The proof is analogous to the proof of Proposition 1, with the opportunity cost equal to β as opposed to 0. □